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EARTHQUAKE STRESSES IN SHEAR BUILDINGS

By M. G. Salvadori, A. M. ASCE

ENGINEERING MECHANICS DIVISION

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PAPERS

EARTHQUAKE STRESSES IN SHEAR BUILDINGS

By M. G. Salvadori,[1] A. M. ASCE

Synopsis

The elastic stresses in a multistory building due to earthquake displacements of its foundation are determined by means of the normal modes of vibration of the building and numerical integration. The building is assumed to move due to the deflection of its columns, and the mass of the structure is concentrated at the floors, which are infinitely rigid. The rocking of the building on an elastic soil and the influence of internal damping are taken into account. The stresses in the multistory building are compared with those obtained by assuming the building to act as a cantilever shear beam. A numerical example illustrates the application of the method and permits an evaluation of its accuracy and efficiency by comparison with a rigorous solution of the same problem.

1. Introduction

The dynamic behavior of an elastic framed building acted upon by impressed displacements (earthquakes) or time-varying forces (blasts) is of great practical importance. Unfortunately, the mathematical difficulties involved in the solution of dynamic problems have prevented a rational design of such structures; hence, any step in the simplification of their solution is to be welcomed.

The dynamical analysis of framed structures is considerably simplified whenever the structure has a height-to-width ratio of 5 or less[2] and whenever its columns are relatively flexible, in which case the structure will be called a "shear building." In a shear building it is possible to neglect the influence of direct stresses in the columns and to assume that: (1) The motion of the building is due to the bending deflection of the columns only; and (2) the mass of the building is lumped at the floor levels. Within the limits of elastic behavior,

Note.—Written comments are invited for publication; the last discussion should be submitted by September 1, 1953.

[1] Associate Prof. of Civ. Eng., Columbia Univ., New York, N. Y., and Cons. Engr., Ammann & Whitney, New York, N. Y.

[2] "Wind Bracing," by H. V. Spurr, McGraw-Hill Book Co., Inc., New York, N. Y., 1930.

the building acts as an elastic system with as many degrees of freedom as there are floors, and the determination of the stresses is reduced to the integration of a set of simultaneous ordinary differential equations.

In what follows, a method for the solution of this problem is presented for the case of earthquake stresses. The same procedures could be applied, practically without change, to the blast problem.

Notation.—Letter symbols in this paper are defined where they first appear, in the text or by diagram. Throughout, the symbol i, used as a subscript, refers the main symbol to its appropriate floor; and the symbol j, used as a subscript, denotes one of the N vibrational modes of the structure. Single or double dots over any symbol, denote differentiation with respect to time.

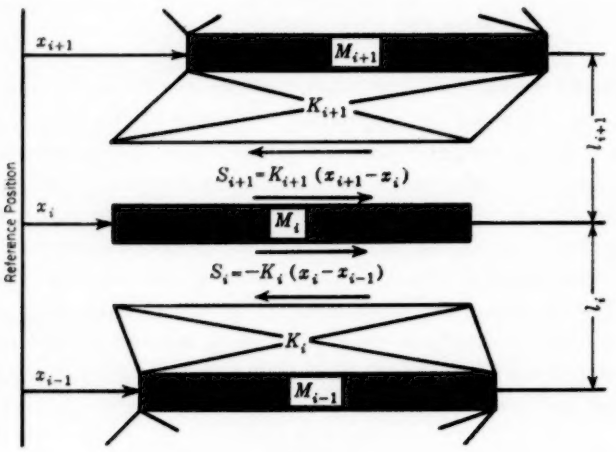

FIG. 1

2. THE FREE VIBRATIONS OF A SHEAR BUILDING

Consider a slice of a framed building, one unit wide, with N floors. In Fig. 1, let:

M_i = the mass concentrated at the ith floor ($i = 0, 1, 2, \ldots, N$);

S_i = the total horizontal shear force at the ith floor;

l_i = the story height between the $(i - 1)$th and the ith floors;

K_i = the spring constant under the ith floor—that is, the shear at the ith floor necessary to displace that floor horizontally by one unit relative to the $(i - 1)$th floor;

$X(i)$ = the modal displacement at the ith floor;

$x_i(t)$ = the absolute horizontal displacement of the ith floor at a time t; and

$x_0(t)$ = the impressed horizontal motion of the foundation of the building, considered identical with the earthquake displacement component in the plane of the frame.

The equation of motion of the ith floor is obtained from Newton's second law ($F = m\,a$) in the form:

$$M_i \ddot{x}_i = K_{i+1}(x_{i+1} - x_i) - K_i(x_i - x_{i-1}) \quad (i = 1, 2, \ldots, N) \ldots (1)$$

in which the dots connote differentiation with respect to time.

The free vibrations of the structure are characterized by the fact that each floor moves in simple harmonic motion. The structure can vibrate freely in N different modes defined by the equation

$$x_i(t) = X(i) \sin \omega t \ldots (2)$$

in which $X(i)$ is the displacement at the ith floor in the mode of frequency ω.[1] Substituting Eqs. 2 in Eqs. 1, the ordinates $X(i)$ are found to satisfy the system of N homogeneous linear equations—

$$-k'_i X(i-1) + (k_i - \omega^2) X(i) - k''_i X(i+1) = 0 \quad (i = 1, 2, \ldots, N) \ldots (3)$$

in which

$$k''_i = \frac{K_{i+1}}{M_i} \ldots (4a)$$

$$k_i = \frac{K_{i+1} + K_i}{M_i} \ldots (4b)$$

and

$$k'_i = \frac{K_i}{M_i} \ldots (4c)$$

Eqs. 3 have roots $X(i)$ different from zero only if their determinant vanishes, thus:

$$\begin{vmatrix} (k_1 - \omega^2) & -k''_1 & 0 & \ldots & 0 & 0 & 0 \\ -k'_2 & (k_2 - \omega^2) & -k''_2 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & & -k'_{N-1} & (k_{N-1} - \omega^2) & -k''_{N-1} \\ 0 & 0 & 0 & & 0 & -k'_N & (k_N - \omega^2) \end{vmatrix} = 0 \ldots (5)$$

The N roots ω^2_j of Eq. 5 ($j = 1, 2, \ldots, N$) are the squares of the frequencies of the N modes. Once the values ω_j have been found, Eqs. 3 determine the "co-ordinates" $X_j(i)$ of the jth mode within a multiplying constant c. This constant may be chosen so as to make

$$\sum_{i=1}^{N} [c^2 X^2_j(i) M_i] \equiv \sum_{i=1}^{N} [\phi^2_j(i) M_i] = 1 \quad (j = 1, 2, \ldots, N) \ldots (6a)$$

in which case the modes

$$\phi_j(i) \equiv c X_j(i) = \frac{X_j(i)}{\sqrt{\sum_{i=1}^{N} X^2_j(i) M_i}} \quad (j = 1, 2, \ldots, N) \ldots (6b)$$

[1] "Mathematical Methods in Engineering," by Th. von Kármán and M. A. Biot, McGraw-Hill Book Co., Inc., New York, N. Y., 1940, Chap. V.

—are said to be normalized. It can be proved[3] that the modes ϕ_j (or X_j) are orthogonal; that is, that

$$\sum_{i=1}^{N} \phi_j(i)\ \phi_k(i)\ M_i = 0 \quad (j \neq k) \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

3. The Forced Vibrations of a Shear Building

The displacements $x_i(t)$ of the structure due to any impressed displacement, or applied force, can be expressed[3] as a series of products of the modes $\phi_j(i)$ and of unknown functions $q_j(t)$, called the "generalized co-ordinates" of the structure. In the study of earthquake displacements it is convenient to expand, in terms of generalized co-ordinates, the displacements of the ith floor relative to the foundation—that is, to set—

$$x_i(t) = x_0(t) + \sum_{j=1}^{N} [\phi_j(i)\ q_j(t)] \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

By substitution of Eqs. 8 in the expressions for the kinetic energy: $T = \frac{1}{2} \sum_{i=1}^{N} (M_i\ \dot{x}^2_i)$, and the potential energy: $U = \frac{1}{2} \sum_{i=1}^{N} [K_i\ (x_i - x_{i-1})^2]$—

of the structure, and by Eqs. 6 and 7, it may be seen that:

$$T = \tfrac{1}{2}\ M\ \dot{x}^2_0 + \dot{x}_0 \sum_{j=1}^{N} (\alpha_j\ \dot{q}_j) + \tfrac{1}{2} \sum_{j=1}^{N} \dot{q}^2_j \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

and

$$U = \tfrac{1}{2} \sum_{j=1}^{N} \omega^2_j\ q^2_j \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

in which M is the total mass of the building and

$$\alpha_j = \sum_{i=1}^{N} [\phi_j(i)\ M_i] \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

The equations of motion of the structure in terms of its generalized co-ordinates q_j are Lagrange's equations[3]

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) + \frac{\partial U}{\partial q_k} = Q_k \quad (k = 1, 2, \ldots, N) \ldots\ldots\ldots\ldots (12a)$$

and

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_0}\right) + \frac{\partial U}{\partial x_0} = Q_0 \ldots\ldots\ldots\ldots\ldots\ldots (12b)$$

where the "generalized force" Q_k is given, in terms of the force P_i applied to the ith floor, by:

$$Q_k = \sum_{i=0}^{N} P_i \frac{\partial x_i}{\partial q_k} \quad (k = 1, 2, \ldots, N) \ldots\ldots\ldots\ldots (13a)$$

$$Q_0 = \sum_{i=0}^{N} P_i \frac{\partial x_i}{\partial x_0} \ldots\ldots\ldots\ldots\ldots\ldots (13b)$$

Eq. 13*b* could be used to determine the force exerted by the ground on the foundation, but will not be considered in this paper. In the present case the only external force is applied at the foundation and $Q_k = P_0 \frac{\partial x_0}{\partial q_k}$; but x_0 is not a function of the q_k and hence all the Q_k are zero. By means of Eqs. 9 and 10, the Lagrangian equations become

$$\ddot{q}_j + \omega^2_j q_j = - \alpha_j \ddot{x}_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

The acceleration graphs supplied by strong-motion records[4] are so jagged that it is often inconvenient to integrate Eqs. 14. Then it becomes practical to introduce new generalized co-ordinates $p_j(t)$ defined by the equation:

$$q_j = \alpha_j (p_j - x_0) \quad \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

in terms of which the equations of motion become

$$\ddot{p}_j + \omega^2_j p_j = \omega^2_j x_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

and contain the displacement x_0, as a forcing function, rather than the acceleration $\ddot{x}_0$. Eqs. 14 and 16 demonstrate the advantage gained by the use of generalized co-ordinates and normal modes. The burdensome integration of the original system of N simultaneous equations (Eqs. 1) has been reduced to the much simpler task of integrating the N separate Eqs. 14 or 16. Moreover, it will often be found sufficient from an engineering point of view to take into account only the first few (two or three) modes of the structure. The problem becomes thus substantially simpler and hence more economical.

Since the motion of the building starts from rest at a time that can be selected as origin ($t = 0$), the initial conditions require that

$$x_i(0) = x_0(0) + \sum_{j=1}^{N} [\phi_j(i)\, q_j(0)] = 0 \quad \ldots\ldots\ldots\ldots (17a)$$

and

$$\dot{x}_i(0) = \dot{x}_0(0) + \sum_{j=1}^{N} [\phi_j(i)\, \dot{q}_j(0)] = 0 \quad \ldots\ldots\ldots\ldots (17b)$$

Multiplying each of Eqs. 17 by $M_i\, \phi_k(i)$, summing with respect to i, and remembering Eqs. 6 and 7:

$$\alpha_k\, x_0(0) + q_k(0) \equiv \alpha_k\, p_k(0) = 0 \quad \ldots\ldots\ldots\ldots\ldots (18a)$$

and

$$\alpha_k\, \dot{x}_0(0) + \dot{q}_k(0) \equiv \alpha_k\, \dot{p}_k(0) = 0 \quad \ldots\ldots\ldots\ldots\ldots (18b)$$

Hence, whatever the values of $x_0(0)$ and $\dot{x}_0(0)$, the initial conditions for p_j are

$$p_j(0) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (19a)$$

$$\dot{p}_j(0) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (19b)$$

[4] "Earthquake Investigations in California 1934–1935," *Special Publication No. 201*, U. S. Dept. of Commerce, Coast and Geodetic Survey, Washington, D. C.

Substituting the value $\dot{q}_k(0) = -\alpha_k \dot{x}_0$ from Eq. 18*b* in Eq. 17*b*, the velocity $\dot{x}_i(0)$ takes the form:

$$\dot{x}_i(0) = \dot{x}_0(0)\left[1 - \sum_{j=1}^{N} \alpha_j \phi_j(i)\right] = 0$$

Whenever $\dot{x}_0(0) \neq 0$ and an approximate solution of the earthquake problem is obtained by means of the first $N' < N$ modes, the ratios—

$$r_{N'}(i) = \frac{\dot{x}_i(0)}{\dot{x}_0(0)} = 1 - \sum_{j=1}^{N'} [\alpha_j \phi_j(i)] \quad \ldots\ldots\ldots\ldots\ldots\ldots (20)$$

—give a quantitative idea of the accuracy with which the initial conditions have been satisfied. (Although $\dot{x}_0(0)$ must vanish for physical reasons, it is sometimes convenient to approximate $x_0(t)$ by means of curves whose first derivative does not vanish at the origin.)

Whenever $x_0(t)$ is an elementary analytic function, the solutions of Eqs. 16 satisfying the initial conditions of Eqs. 19 are given by

$$p_j(t) = \omega_j \int_0^t x_0(\tau) \sin \omega_j (t - \tau)\, d\tau \quad \ldots\ldots\ldots\ldots\ldots\ldots (21)$$

For example, if

$$x_0(t) = X_0 \sin \omega_0 t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

Eq. 21 gives

$$p_j(t) = \frac{X_0 \omega_j}{\omega^2_j - \omega^2_0} (\omega_j \sin \omega_0 t - \omega_0 \sin \omega_j t); \quad (\omega_0 \neq \omega_j) \ldots\ldots (23a)$$

and

$$p_k(t) = \tfrac{1}{2} X_0 (\sin \omega_0 t - \omega_0 t \cos \omega_0 t); \quad (\omega_0 = \omega_k) \ldots\ldots\ldots (23b)$$

and the corresponding displacements become, by Eqs. 8:

$$x_i(t) = X_0 \left[1 + \omega^2_0 \sum_{j=1}^{N} \frac{\alpha_j \phi_j(i)}{\omega^2_j - \omega^2_0}\right] \sin \omega_0 t - X_0 \omega_0 \sum_{j=1}^{N} \frac{\alpha_j \omega_j \phi_j(i)}{\omega^2_j - \omega^2_0} \sin \omega_j t; \quad (\omega_j \neq \omega_0) \ldots (24a)$$

and

$$x_i(t) = X_0 \left[1 + \omega^2_0 \sum_{\substack{j=1 \\ j \neq k}}^{N} \frac{\alpha_j \phi_j(i)}{\omega^2_j - \omega^2_0}\right] \sin \omega_0 t - X_0 \omega_0 \sum_{\substack{j=1 \\ j \neq k}}^{N} \frac{\alpha_j \omega_j \phi_j(i)}{\omega^2_j - \omega^2_0} \sin \omega_j t - \tfrac{1}{2} X_0 \alpha_k \phi_k(i) [\sin \omega_0 t + \omega_0 t \cos \omega_0 t]; \quad (\omega_0 = \omega_k) \ldots (24b)$$

By superposition, the displacements due to a half-sine pulse ground displacement—

$$x_0(t) = \begin{cases} X_0 \sin \omega_0 t; & 0 < t \leq \dfrac{\pi}{\omega_0} \\ 0 \qquad\qquad ; & \dfrac{\pi}{\omega_0} \leq t \end{cases} \quad \ldots\ldots\ldots\ldots\ldots (25)$$

—are given by

$$x_{i,p}(t) = \begin{cases} x_i(t) & ; \quad 0 < t \leq \dfrac{\pi}{\omega_0} \\ x_i(t) + x_i\left(t - \dfrac{\pi}{\omega_0}\right); & \dfrac{\pi}{\omega_0} \leq t \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots (26)$$

The solutions of Eqs. 24 may be used in connection with each sine component of the expansion of $x_0(t)$ into a Fourier series and are of practical importance in connection with the graph of maximum amplitude as functions of the frequency given by Frank Neumann,[5] for each grade of the modified Mercalli earthquake scale.

When $x_0(t)$ is not an elementary function, and particularly when $x_0(t)$ is known experimentally, the integration of Eqs. 16 is conveniently performed by numerical methods, as shown in Sect. 8, or by analog machines.

4. The Influence of Internal Damping

Although little is known about the type of internal damping developed in actual structures, experimental evidence shows[6] that damping in steel occurs as if the coefficient of viscous damping in any given mode were proportional to the frequency of that mode. Since viscous damping is easily handled mathematically, this result is assumed to apply to a complete structure, and a viscous term is added to Eqs. 16:

$$\ddot{p}_j + 2 f_j \dot{p}_j + \omega^2_j p_j = \omega^2_j x_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (27)$$

The value of the logarithmic decrement in the jth mode—

$$r_j = e^{-2\pi f_j/\omega_j} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28)$$

—has an upper value (seldom attained) of 0.9; and, hence, an upper value for the coefficient of viscous damping is

$$f_j \leq - \frac{\log_e 0.9}{2\pi} \omega_j = 0.017\, \omega_j \quad \ldots\ldots\ldots\ldots\ldots\ldots (29)$$

The solution of Eqs. 27 when $x_0(t)$ is a sine function is

$$x_i(t) = X_0 \{\sin \omega_0 t + e^{-f_j t} [A_j \cos \sqrt{\omega^2_j - f^2_j}\, t + B_j \sin \sqrt{\omega^2_j - f^2_j}\, t] + C_j \cos \omega_0 t + D_j \sin \omega_0 t\} \quad \ldots (30)$$

in which

$$A_j = - C_j = \frac{\alpha_j \omega^2_0 + 2 f_j \omega_0}{(\omega^2_j - \omega^2_0)^2 + (2 f_j \omega_0)^2} \quad \ldots\ldots\ldots\ldots\ldots (31a)$$

$$B_j = - \frac{\alpha_j \omega_0}{\sqrt{\omega^2_j - f^2_j}} \frac{\omega^2_j (\omega^2_j - \omega^2_0) + 2 f^2_j \omega^2_j}{(\omega^2_j - \omega^2_0)^2 + (2 f_j \omega_0)^2} \quad \ldots\ldots\ldots\ldots\ldots (31b)$$

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[5] "Ground Amplitudes and Periods Associated With Earthquake Intensity: A Preliminary Statement," by Frank Neumann, Supplement, *Quarterly Engineering Seismology Bulletin*, MST-75, October, November, December, 1950.

[6] "Response of Damped Elastic Systems to Transient Disturbances," by R. D. Mindlin, F. W. Stubner, and H. L. Cooper, *Bell Telephone System Tech. Publications*, Monograph B 1561 (or *Proceedings*, Soc. for Experimental Stress Analysis, Vol. 5, No. 2, 1948.)

and

$$D_j = \frac{\alpha_j \omega^2_0 (\omega^2_j - \omega^2_0)}{(\omega^2_j - \omega^2_0)^2 + (2 f_j \omega_0)^2} \quad \ldots\ldots (31c)$$

Inasmuch as the largest stresses due to earthquakes occur, in general, a short time after the beginning of the earthquake, and since the logarithmic decrement r_j is nearly 1, the influence of damping on elastic earthquake stresses is often negligible. Its essential feature is to damp out, rapidly, the higher modes of vibration.

5. Rocking of Building on an Elastic Soil

It was assumed in the preceding sections that the motion of the foundation was purely horizontal. When a building rests on an elastic soil, instead, the earthquake displacements will produce, not only horizontal displacements of the floors, but also rotation of the foundation about an axis through its centroid perpendicular to the plane of motion. (It is assumed that no relative horizontal displacement takes place between foundation and soil.) This additional motion is usually called the "rocking" of the building.

Calling θ the rotation of the foundation in radians, indicate by B (elastic stiffness coefficient for rocking motion) the reactive elastic moment of the soil per unit depth of the building when $\theta = 1$. The value of B was computed by M. A. Biot[7] in terms of the elastic constants E(Young's modulus) and ν (Poisson's ratio) of the soil:

$$B = \frac{\pi}{4} \frac{E}{1 - \nu^2} \left(\frac{b}{2}\right)^2 \quad \ldots\ldots (32)$$

in which b is the width of the foundation.

The modes of the elastic structure rocking on its foundation are conveniently determined by means of the concept of "complementary systems" evolved by Hans H. Bleich, M. ASCE.[8] As "complementary system A," consider the building in which the "shear springs" are frozen: System A is a rigid beam elastically restrained against rotation at one end. As "complementary system B," consider the building in which the "shear springs" are free to act, but in which rocking is prevented. The modes ϕ_j of system B are, therefore, the modes, previously determined, of the building on a rigid soil.

The differential equation of free vibration of the rigid building (system A) is

$$\ddot{\theta} + \omega^2_A \theta = 0 \quad \ldots\ldots (33)$$

in which the frequency ω_A is given by

$$\omega_A = \sqrt{\frac{B}{I}} \quad \ldots\ldots (34)$$

and

$$I = \sum_{i=0}^{N} M_i \left[L^2_i + \frac{b^2}{12} \right] \quad \ldots\ldots (35)$$

[7] "Analytical and Experimental Methods in Engineering Seismology," by M. A. Biot, *Transactions*, ASCE, Vol. 108, 1943, p. 365.

[8] "Frequency Analysis of Beam and Girder Floors," by Hans H. Bleich, *Transactions*, ASCE, Vol. 115, 1950, p. 1023.

is the moment of inertia of the structure about the axis of rotation, L_i indicating the height of the ith floor above the foundation (Fig. 2). The corresponding mode $\phi_A(i)$ is a straight line

$$\phi_A(i) = \frac{L_i}{\sqrt{\sum_{i=1}^{N} M_i L^2_i}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (36)$$

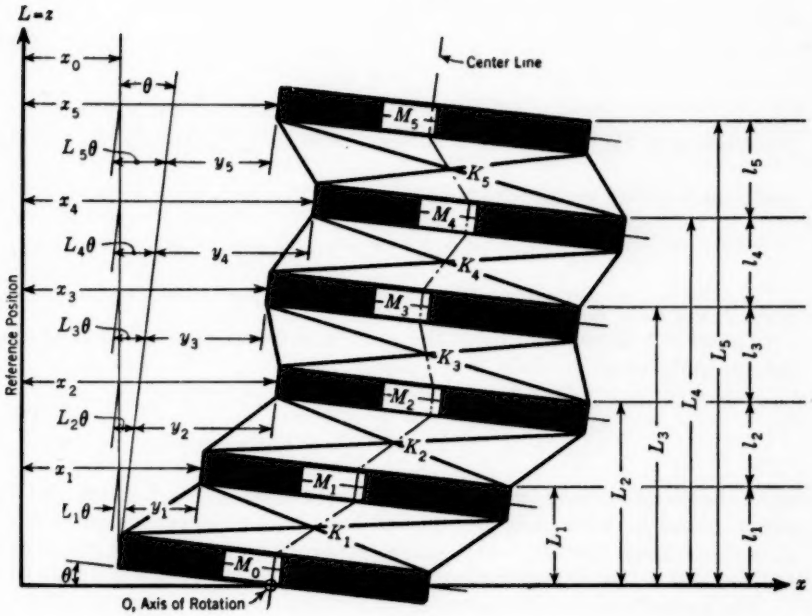

FIG. 2

Mr. Bleich[8] has shown that the modes $\psi_k(i)$ of the rocking structure can be represented by a linear combination of $\phi_A(i)$ and $\phi_j(i)$, thus:

$$\psi_k(i) = c_k \phi_A(i) + \sum_{j=1}^{N} c_{kj} \phi_j(i); \quad (k = 1, 2, \ldots, N+1) \ldots\ldots (37)$$

The coefficients of this linear combination are given by

$$\frac{\omega^2_A}{\Omega^2_k} c_k = \sum_{i=1}^{N} [\phi_A(i) \psi_i(i) M_i] \ldots\ldots\ldots\ldots\ldots\ldots (38)$$

and

$$\frac{\omega^2_j}{\Omega^2_k} c_{kj} = \sum_{i=1}^{N} [\phi_j(i) \psi_k(i) M_i] \ldots\ldots\ldots\ldots\ldots\ldots (39)$$

in which Ω_k is the frequency of the mode ψ_k. Substituting in Eqs. 38 and 39

the value of ψ_k defined by Eq. 37 and letting

$$b_j = \sum_{i=1}^{N} M_i \, \phi_A(i) \, \phi_j(i) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (40)$$

it follows that

$$c_k \left(\frac{\omega^2_A}{\Omega^2_k} - 1 \right) = \sum_{j=1}^{N} b_j \, c_{kj} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (41a)$$

and

$$c_{kj} = \frac{b_j}{\dfrac{\omega^2_j}{\Omega^2_k} - 1} \, c_k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (41b)$$

Substitution of Eq. 37 in the normalizing condition $\sum_{i=1}^{N} [\psi^2_k \, (i) M_i] = 1$ shows that c must satisfy the condition

$$\omega^2_A \, c^2_k + \sum_{j=1}^{N} \omega^2_j \, c^2_{kj} = \Omega^2_k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (42)$$

if the ψ_k are to be normal modes.

The $N + 1$ frequencies Ω_k of the rocking structure are the roots of the determinantal equation of the $N + 1$ Eqs. 41:

$$\frac{\omega^2_A}{\Omega^2_k} - 1 = \sum_{j=1}^{N} \frac{b^2_j}{\dfrac{\omega^2_j}{\Omega^2_k} - 1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (43)$$

Once the Ω_k values have been found from Eq. 43, the $N + 1$ Eqs. 41*b* and 42 define c_k and c_{kj} for each value of k, and Eq. 37 gives the new normal modes ψ_k of the rocking structure. (An analysis of Eq. 43 shows that Ω_1 is lower than both ω_A and ω_1, and that any other Ω_k is bound by two successive ω_j.)

The generalized co-ordinates p_k of the rocking structure satisfy the first or second of the equations

$$\ddot{p}_k + \Omega^2_k \, p_k = \Omega^2_k \, x_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (44a)$$

and

$$\ddot{p}_k + 2 f_k \, p_k + \Omega^2_k \, p_k = \Omega^2_k \, x_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (44b)$$

depending upon whether friction is neglected or taken into account; and the initial conditions are

$$p_k(0) = 0; \quad \text{and} \quad \dot{p}_k(0) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (45)$$

The absolute displacements of the rocking structure are given, as usual, by

$$x_i(t) = x_0(t) + \sum_{k=1}^{N+1} \psi_k(i) \, q_k(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (46)$$

in which, as in Eq. 15, $q_k = \alpha_k(p_k - x_0)$.

It is now interesting to notice that the stresses in the rocking building are due exclusively to the shear components of the displacements x_i. These are obtained by subtracting from the total modes ψ_k the rotational components

$c_k\,\phi_A$ to obtain the "shear modes,"

$$\psi_{k,s}(i) = \sum_{j=1}^{N} [c_{kj}\,\phi_j(i)] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (47)$$

The shear displacements are then simply given by

$$x_{i,s}(t) = \sum_{k=1}^{N+1} \psi_{k,s}(i)\,q_k(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (48)$$

6. The Numerical Evaluation of the Modes

When the number of stories N of the building is greater than 4 or 5, the writing and the solution of the determinantal equation for the frequencies (Eq. 5) becomes a lengthy task requiring particular care. It is simpler in such cases to obtain the frequencies and the modes by the classical procedure of successive approximations developed by L. Vianello and A. Stodola, which makes use of the fact that the modes are the deflections of the elastic structure loaded by its own inertia forces.

To determine the first mode $X_1(i)$ by the Vianello-Stodola method:

(1) Assume a deflection $X_1^{(0)}(i)$, which possibly is not too different from $X_1(i)$;

(2) Apply to the structure the horizontal loads $P_i = \omega^2_1\,M_i\,X_1^{(0)}(i)$;

(3) Determine the shear S_k at the kth floor: $S_k = \sum_{i=N}^{k} P_i$

(4) Determine the displacement of the kth floor, relative to the $(k-1)$th floor: $\Delta X_k = S_k/K_k$ (in which K_k is the spring constant under the kth floor)

(5) Determine the new displacement $X_1^{(1)}(i)$ of the ith floor by summing the ΔX_k from the foundation up: $X_1^{(1)}(i) = \sum_{k=1}^{i} \Delta X_k$.

The displacement $X_1^{(1)}$ is a better approximation of X_1 which can now be used as the initial deflection to obtain a nearer approximation $X_1^{(2)}$, and the process is continued until the difference between two successive approximations becomes negligible.

In practice, since ω^2_1 is unknown, the structure is loaded with loads $P_i = M_i\,X_1^{(0)}(i)$ and the deflections $X_1^{(1)}(i)$ are ω^2_1 times smaller than they should be. Hence the ratio of any $X_1^{(r)}(i)$ to the corresponding $X_1^{(r+1)}(i)$ gives an approximate value of ω^2_1.

The best possible value of ω^2_1 is obtained by taking the ratio of the weighted averages of $X_1^{(r)}$ and $X_1^{(r+1)}$,

$$\omega^4_1 = \frac{\sum_{i=1}^{N} [X_1^{(r)}(i)]^2\,M_i}{\sum_{i=1}^{N} [X_1^{(r+1)}(i)]^2\,M_i} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (49)$$

The normalized mode $\phi_1(i)$ is given, in terms of the last approximation $X_1^{(n)}(i)$ by Eq. 6*b*.

The second mode ϕ_2 may be obtained by the same procedure, provided the initial approximation $X_2^{(0)}$ is chosen orthogonal to ϕ_1 and the successive approximations $X_2^{(r)}$ are made orthogonal to ϕ_1. It is easy to see that, given any function $\overline{X}(i)$, the function—

$$X(i) = \overline{X}(i) - \phi_1(i) \sum_{i=1}^{N} [\overline{X}(i)\, \phi_1(i)\, M_i] \quad \ldots\ldots\ldots\ldots (50)$$

—is orthogonal to ϕ_1. Hence, X may be chosen as $X_2^{(0)}$, and Eq. 50 may be used to obtain functions $X_2^{(r)}$ orthogonal to ϕ_1, by means of the rth approximation $\overline{X}_2^{(r)}$, which, in general, will not be orthogonal to ϕ_1.

The determination of the third mode may similarly be started by means of any function $\overline{X}$ and of the formula

$$X(i) = \overline{X}(i) - \phi_1(i) \sum_{i=1}^{N} [\overline{X}(i)\, \phi_1(i)\, M_i] - \phi_2(i) \sum_{i=1}^{N} [\overline{\overline{X}}(i)\, \phi_2(i)\, M_i] \ldots (51)$$

which makes X orthogonal to both ϕ_1 and ϕ_2.

In order to be able to follow the convergence of the method it is practical to normalize the approximations $X_i^{(r)}$ at each step. Since the results converge more rapidly when the initial approximations are near the true modes, it pays to obtain good initial guesses. These may be obtained, for example, from the modes of a "shear beam."

7. The Modes and Frequencies of the Shear Beam

Refer the building on a rigid soil to a vertical z-axis with the origin at the foundation (Fig. 2), and let:

$z_i = L_i$, the height of the ith floor above the foundation;

$l_i = z_i - z_{i-1}$, the height of the ith story;

$m = \dfrac{M_i}{l_i}$, the average mass per unit length (assumed to be constant);

$k = K_i\, l_i$, the average spring constant per unit length (assumed to be constant);

$L = \sum_{i=1}^{N} l_i$, the height of the building;

$M = \sum_{i=1}^{N} M_i$, the total mass of the building; and

$\overline{K}$ = the common value of K_i, when K_i does not vary from story to story.

Letting

$$x_i(t) = x(z, t); \quad x_{i+1} - x_i = \Delta x; \quad z_i - z_{i-1} = \Delta z \quad \ldots\ldots\ldots (52)$$

—Eqs. 1 take the form:

$$m\,\Delta z\,\ddot{x} = \Delta\left(\frac{k}{\Delta z}\Delta x\right) = \frac{k}{\Delta z}\Delta^2 x$$

and, dividing by Δz and taking the limit as Δz approaches zero, the partial differential equation of a so-called "shear beam" becomes

$$\frac{\partial^2 x}{\partial t^2} = \frac{k}{m}\frac{\partial^2 x}{\partial z^2} \quad \ldots\ldots (53)$$

In this equation k and m (and hence their ratio k/m) are assumed to be independent of z. (The solution of the corresponding equation for a linear variation of k/m has been obtained by the writer in connection with further studies on the dynamics of shear buildings.[9]) When k/m is constant, it is easy to prove[10] that the normal modes of the shear beam are given by

$$\phi_n(z) = \sqrt{\frac{2}{m\,L}}\sin\,(2\,n-1)\,\frac{\pi\,z}{2\,L};\quad (n = 1, 2, 3, \ldots) \quad \ldots\ldots (54)$$

and that the corresponding frequencies are given by

$$\omega_n = (2\,n-1)\,\frac{\pi}{2\,L}\sqrt{\frac{k}{m}};\quad (n = 1, 2, 3, \ldots) \quad \ldots\ldots (55)$$

Merit P. White,[11] M. ASCE, has proved, moreover, that as soon as the number of floors N is larger than 3, the first few frequencies, and, hence, the first few modes, of the lumped building, for all practical purposes, are sufficiently well approximated by those of the uniform shear beam.

Hence, when the ratios $\frac{k}{m} = \frac{K_i\,l^2_i}{M_i}$ are approximately constant, one may obtain good approximations to the lower frequencies and modes of the building by using for k, in Eq. 55, its average value—

$$k \doteq \frac{1}{N}\sum_{i=1}^{N} K_i\,l_i \quad \ldots\ldots (56)$$

—and for m in Eqs. 54 and 55 its average value

$$m \doteq \frac{M}{L} \quad \ldots\ldots (57)$$

obtaining

$$\phi_j(i) \doteq \sqrt{\frac{2}{M}}\sin\,(2\,j-1)\,\frac{\pi}{2}\,\frac{L_i}{L} \quad \ldots\ldots (58)$$

and

$$\omega_j \doteq (2\,j-1)\,\frac{\pi}{2}\sqrt{\frac{\sum_{i=1}^{N}(K_i\,l_i)}{N\,M\,L}} \quad \ldots\ldots (59)$$

[9] "Approximate Determination of Dynamic Stresses in Buildings," by M. G. Salvadori, *Proceedings* Eighth International Cong. on Theoretical and Applied Mechanics, Istanbul, Turkey, 1952.

[10] "Natural Periods of Uniform Cantilever Beams," by L. S. Jacobsen, *Transactions*, ASCE, Vol. 104, 1939, p. 402.

[11] *Ibid.* p. 433.

Moreover, if M_i, K_i and l_i are independently constant (uniform building of constant story height), Eqs. 58 and 59 reduce to:

$$\phi_j(i) \doteq \sqrt{\frac{2}{M}} \sin (2j - 1) \frac{i}{N} \frac{\pi}{2} \quad \ldots\ldots (60a)$$

and

$$\omega_j \doteq (2j - 1) \frac{\pi}{2} \sqrt{\frac{\overline{K}}{N M}} \quad \ldots\ldots (60b)$$

As shown by the illustrative example in Sect. 9, these formulas give fairly good approximations to the lower modes and frequencies of a building even if $\frac{K_i}{M_i} l^2_i$ is not rigorously constant.

8. The Numerical Integration of the Equations of Motion

As pointed out in Sect. 3, the equations of motion (Eqs. 16 or 27) must be integrated numerically whenever $x_0(t)$ is not a simple analytical function. Numerous methods of step-by-step integration are available for second-order equations, but few are as simple and as powerful as those advanced by Noumerov, and by L. Fox and R. T. Goodwin,[12] and applicable to Eqs. 16 and 27.

To integrate Eq. 16

$$\ddot{p} + \omega^2 p = \omega^2 x_0 \quad \ldots\ldots (61)$$

Eq. 61 is multiplied through by the square of the time spacing $h = \Delta t$, the second derivative is approximated by the first two terms $\left(\delta^2 - \frac{\delta^4}{12}\right) p$ of its central difference expansion[13,14] and both sides of the equation are then operated upon by $\left(1 + \frac{\delta^2}{12}\right)$. Letting

$$p(rh) = p_r \quad \ldots\ldots (62a)$$

$$\omega^2_0 \, x_0(rh) = f_r \quad \ldots\ldots (62b)$$

$$\delta^2 p_r = p_{r+1} - 2 p_r + p_{r-1} \quad \ldots\ldots (62c)$$

$$c = \frac{2 - \frac{5}{6} h^2 \omega^2}{1 + \frac{1}{12} h^2 \omega^2} \quad \ldots\ldots (62d)$$

and

$$F_r = \frac{h^2}{12} \frac{f_{r+1} + 10 f_r + f_{r-1}}{1 + \frac{1}{12} h^2 \omega^2} \quad \ldots\ldots (62e)$$

—Eq. 61 reduces to the simple recurrence equation

$$p_{r+1} = - p_{r-1} + c \, p_r + F_r + e_{r+1} \quad \ldots\ldots (63)$$

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[12] "Some New Methods for the Numerical Integration of Ordinary Differential Equations," by L. Fox and R. T. Goodwin, *Proceedings*, Cambridge Physical Soc., Vol. 45, Part 3, 1949, p. 373.

[13] "Central Difference Formulae," by W. F. Sheppard, *Proceedings*, London Mathematical Soc., Vol. 31, 1899.

[14] "Numerical Computation of Buckling Loads by Finite Differences," by M. G. Salvadori, *Transactions*, ASCE, Vol. 116, 1951, p. 590.

in which the truncation error e_{r+1} takes the form

$$e_{r+1} = \frac{-1}{1 + \frac{1}{12} h^2 \omega^2} \left(\frac{\delta^6}{240} - \frac{13\,\delta^8}{13{,}120} + \dots \right) p_r \dots\dots\dots (64)$$

The truncation error is of the order of h^6 and the convergence of the process is very rapid.

In order to be able to apply Eq. 63, p must be known at $r = 0$ and $r = 1$. Although p_1 could be evaluated by means of a Taylor expansion around $t = 0$, due to the uncertainties in the value of x_0 and its derivatives at $t = 0$, it is more convenient and accurate to evaluate p_1 by means of the initial conditions expressed in terms of averaged central differences. Neglecting differences of order higher than the fourth, as was done in obtaining Eq. 63, Eqs. 19 yield

$$p(0) = p_0 = 0 \dots\dots\dots (65a)$$

and

$$\dot{p}(0) \doteq \mu \left(\delta - \frac{\delta^3}{6} \right) p_0$$

$$= \frac{1}{2h} \left[p_1 - p_{-1} - \tfrac{1}{6} (p_2 - 2\,p_1 + 2p_{-1} - p_{-2}) \right] = 0 \dots (65b)$$

whereas, by applying Eq. 63 at $r = -1, 0, 1$, and noticing that $F_{-1} = 0$:

$$c\,p_{-1} - p_{-2} = 0 \dots\dots\dots (66a)$$

$$p_1 + p_{-1} = F_0 \dots\dots\dots (66b)$$

and

$$p_2 - c\,p_1 = F_1 \dots\dots\dots (66c)$$

Solving Eqs. 66 and Eq. 65a for p_1 and p_2:

$$p_1 = \tfrac{1}{2} \left(F_0 + \frac{F_1}{8 - c} \right) \dots\dots\dots (67a)$$

and

$$p_2 = \tfrac{1}{2} \left(c\,F_0 + \frac{16 - c}{8 - c} F_1 \right) \dots\dots\dots (67b)$$

and the application of Eq. 63 is begun at $r = 2$, neglecting the error e_{r+1}. Once p_r is evaluated at the required points, e_{r+1} may be obtained by differencing the values p_r for $r = 3, 4, 5, \dots$ and by extrapolation at sight for $r = 1, 2, 3$. These values are introduced in Eq. 63, at each step, to obtain improved values of p_r. It is seldom that this correction need be used in practical cases.

Substituting for the derivatives the first term of their central difference expansions,[13,14] the following recurrence formula for the integration of Eqs. 27 is similarly obtained:

$$p_{r+1} = -c_1\,p_{r-1} + c_2\,p_r + F_r + e'_{r+1} \dots\dots\dots (68)$$

in which

$$c_1 = \frac{1 - hf}{1 + hf} \quad \ldots\ldots (69a)$$

$$c_2 = \frac{2 - h^2\omega^2}{1 + hf} \quad \ldots\ldots (69b)$$

$$F_r = \frac{\omega^2 x_0(rh)}{1 + hf} \quad \ldots\ldots (69c)$$

and

$$e'_{r+1} = \frac{1}{1 + hf}\left[\left(\frac{\delta^4}{12} - \frac{\delta^6}{90} + \ldots\right) p_r + 2fh\mu\left(\frac{\delta^3}{6} - \frac{\delta^5}{30} + \ldots\right) p_r\right] \ldots (69d)$$

In Eq. 69*d*, f is the coefficient of viscous friction. The initial values p_0 and p_1 are both equal to zero in this case, and the truncation error is proportional to h^4.

9. Illustrative Example

a. Modes and Frequencies of Building on Rigid Soil.—The methods presented in the previous section will be illustrated by means of the simple example of a five-story building, whose geometric and elastic characteristics are given in Table 1.

TABLE 1.—Geometric and Elastic Characteristics of a Five-Story Building on Rigid Soil

Floor i (1)	Height l_i $\times 10^{-2}$ in. (2)	Height L_i $\times 10^{-2}$ in. (3)	$M_i \times \left(\frac{\text{lb-sec}^2}{\text{in.}}\right)$ (4)	$K_i \times 10^{-4} \frac{\text{lb}}{\text{in.}}$ (5)	$-k''_i$ (6)	k_i (7)	$-k'_i$ (8)	$\frac{K_i l_i^2}{M_i} \times 10^{-4}$ (9)
5	1.44	7.44	11.65	6.84	—	587	−587	12.17
4	1.44	6.00	15.53	7.98	−440	954	−514	10.66
3	1.44	4.56	15.53	9.12	−514	1,101	−587	12.18
2	1.44	3.12	15.53	10.26	−587	1,248	−661	13.70
1	1.68	1.68	15.53	11.40	−661	1,395	—	20.72
0	—	—	62.08	—	—	—	—	—

$L = \sum_{i=1}^{5} l_i = 744$ in.; and $M = \sum_{i=1}^{5} M_i = 73.77$ lb-sec^2 per in.

The roots of the corresponding determinantal equation, Eq. 5, are given in Table 2; they were evaluated by Newton's method and synthetic division in about one hour. Solving Eqs. 3 successively with these values of ω_j by taking $X_j(5) = 1$, the $X_j(i)$ are obtained; and, by Eq. 6*b*, the modes $\phi_j(i)$ are found and entered in Table 3(*a*).

TABLE 2.—Roots of Equation 5, Five-Story Building

j	ω^2_j	ω_j	ω_j of Shear Beam	ϵ%
1	57.11	7.557	7.84	3.7
2	424.89	20.61	23.53	14.1
3	1,009.73	31.78	39.21	23.3
4	1,596.19	39.95	54.90	37.4
5	2,197.09	46.87	70.58	50.5

TABLE 3.—COMPARISON OF MODES COMPUTED BY EQUATIONS 6*b* AND 58, FOR SEVERAL VALUES OF j

Floor i	(*a*) MODES $\phi_j(i)$, BY EQUATION 6*b*					(*b*) APPROXIMATE MODES ϕ_j BY SHEAR BEAM (Eq. 58)				
	Values of j equal to:									
	1	2	3	4	5	1	2	3	4	5
5	0.165	−0.166	0.149	−0.093	0.020	0.165	−0.165	0.165	−0.165	0.165
4	0.149	−0.046	−0.107	0.160	−0.055	0.157	−0.101	0.009	0.087	−0.151
3	0.118	0.095	−0.116	−0.120	0.116	0.135	0.041	−0.164	0.072	0.114
2	0.080	0.149	0.076	−0.039	−0.168	0.101	0.151	−0.025	−0.164	−0.057
1	0.040	0.102	0.130	0.127	0.139	0.057	0.144	0.161	0.101	−0.008

The uniformly distributed mass and spring constant of the five-story building are, by Eqs. 57 and 56, respectively, $m = 0.0992$ lb-sec^2 per sq in., and $k = 1.37 \times 10^6$ lb. The frequencies of the first five modes of the corresponding shear beam, computed from Eq. 59, appear in Table 2 together with their percentage errors. They show that, although the ratios $K_i l^2_i/M_i$ (Table 1) are not constant, the first frequency is fairly well approximated by the fundamental frequency of the shear beam. The modes of the shear beam, computed by Eq. 58, appear for comparison in Table 3(*b*).

Table 4, which is self-explanatory, shows the evaluation of the first mode by the Stodola-Vianello method, according to Sect. 6, starting from an assumed deflection $X_1^{(0)}$. The second approximation, $\phi_1^{(2)}(i)$, has errors of not more than 1 unit in the third significant figure, when compared with the true mode ϕ_1 of Table 3(*a*). The corresponding frequency, evaluated by Eq. 49, is $\omega^4_1 \doteq \dfrac{0.9217}{0.0002823} = 3{,}265$; and $\omega_1 \doteq 7.559$—and has an error of less than 0.03%.

The evaluation of the second mode by the Stodola-Vianello method appears in Table 5: The assumed deflection $\overline{X}_2$ is normalized by means of Eq. 6 to obtain $\overline{\phi}_2^{(0)}$; and $\overline{\phi}_2^{(0)}$ is made orthogonal to ϕ_1 by means of Eq. 50 to obtain the initial approximation $\phi_2^{(0)}$. The first approximation $X_2^{(1)}$ is then obtained by the same operations used in Table 4 to derive $X_1^{(1)}$ from $X_1^{(0)}$. The displacement $X_2^{(1)}$ is normalized, $\overline{\phi}_2^{(1)}$, and made orthogonal to ϕ_1, $\phi_2^{(1)}$. In this example $\overline{\phi}_2^{(1)}$ is already orthogonal to ϕ_1 and hence $\overline{\phi}_2^{(1)} = \phi_2^{(1)}$. (This indicates that, when the approximate deflection is near the true mode, one need not orthogonalize the solution at each step.) Repeating the process, $\phi_2^{(2)}$ is found to differ from the true ϕ_2 of Table 3(*a*) by no more than three units in the third significant figure—that is, by less than 3%. The corresponding value of the frequency after the second approximation is $\omega^4_2 \doteq \dfrac{\sum_{i=1}^{N} [\phi_2^{(1)}(i)]^2 M_i}{\sum_{i=1}^{N} [X_2^{(2)}(i)]^2 M_i} = \dfrac{1}{5.426 \times 10^{-6}}$; and, $\omega_2 \doteq 20.72$—with an error of 0.53%; and, after the third approximation, $\omega_2 = 20.63$, with an error of 0.05%.

TABLE 4.—EVALUATION OF THE FIRST

i	M_i	$X_1^{(0)}$	$10^{-3}\, M_i X_1^{(0)}$	$10^{-3}\, S_i^{(0)}$	$10^{-3}\, K_i$	$\Delta X_1^{(1)}(i) = \frac{S_i^{(0)}}{K_i}$	$X_1^{(1)}$	$M_i X_1^{(1)}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
5	11.65	9	0.105	0.105	6.84	0.0153	0.158	1.836
4	15.53	8	0.124	0.229	7.98	0.0287	0.142	2.210
3	15.53	6.5	0.101	0.330	9.12	0.0362	0.114	1.765
2	15.53	4.5	0.070	0.400	10.26	0.0390	0.078	1.203
1	15.53	2.5	0.039	0.439	11.40	0.0385	0.039	0.598

TABLE 5.—EVALUATION OF THE SECOND

i	$\bar{X}_2$	$\bar{\phi}_2^{(0)}$	$\phi_2^{(0)}$	$M_i \phi_2^{(0)}$	$S_i^{(0)}$	$10^3\, \Delta X_2^{(1)}$	$10^3\, X_2^{(1)}$	$\bar{\phi}_2^{(1)}$	$\phi_2^{(1)}$	$M_i \phi_2^{(1)}$	$S_i^{(1)}$	$10^3\, \Delta X_2^{(2)}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
5	−10	−0.153	−0.152	−1.767	−1.767	−0.258	−0.338	−0.159	−0.159	−1.852	−1.852	−0.271
4	0	0	0.001	0.018	−1.749	−0.219	−0.080	−0.038	−0.038	−0.582	−2.435	−0.305
3	0	0	0.001	0.014	−1.734	−0.190	0.139	0.066	0.066	1.019	−1.416	−0.155
2	10	0.153	0.154	2.386	0.652	0.064	0.330	0.155	0.155	2.409	0.993	0.097
1	10	0.153	0.153	2.381	3.033	0.266	0.266	0.125	0.125	1.944	2.937	0.258

The third mode was not computed since it did not contribute substantially to the solution, as shown in Sect. 9c.

b. Modes and Frequencies of Rocking Building.—A soft elastic soil with $E = 10{,}000$ lb per sq in., and Poisson's ratio $\nu = \frac{1}{3}$, has an elastic stiffness coefficient for the rocking motion of a unit strip of a building 60 ft wide, given by Eq. 32; thus: $B = 1.145 \times 10^9$ in.-lb. The moment of inertia of the building, about an axis through the centroid of its foundation, by Eq. 35, with M_i and L_i as in Table 1 and $b = 60$ ft, is $I = 2.308 \times 10^7$ lb-in.-sec^2. Hence, the frequency of the rigid building, by Eq. 34, is $\omega_A = 7.043$ ($\omega^2_A = 49.60$). The corresponding mode ϕ_A, computed by Eq. 36, appears in Table 6 together with the first two modes ϕ_1 and ϕ_2 of the shear building on a rigid soil, taken from Table 3(*a*).

The modes of the rocking structure have been computed approximately by means of these three modes only. From Eq. 40 and Table 6, $b_1 = 0.998$; and $b_2 = -0.049$; and with $\omega^2_1 = 57.11$, $\omega^2_2 = 424.9$ (from Table 2) the frequency

TABLE 6.—MODES OF ROCKING STRUCTURE

i	$\phi_A(i)$	$\phi_1(i)$	$\phi_2(i)$	$\psi_1(i)$	$\psi_{1,e}(i)$	$\psi_2(i)$	$\psi_{2,e}(i)$
5	0.179	0.165	−0.166	0.173	0.077	0.162	0.129
4	0.145	0.149	−0.046	0.147	0.069	0.040	0.014
3	0.110	0.118	0.095	0.114	0.055	−0.099	−0.119
2	0.075	0.080	0.149	0.077	0.037	−0.151	−0.165
1	0.041	0.040	0.102	0.040	0.018	−0.102	−0.109

MODE BY THE STODOLA–VIANELLO METHOD

$M_i[X_1^{(1)}(i)]^2$	$S_i^{(1)}$	$10^2\,\Delta X_1^{(2)}$	$10^2\,X_1^{(2)}(i)$	$10^4[X_1^{(2)}]^2$	$10^4 M_i[X_1^{(2)}]^2$	$\phi^{(2)}{}_1(i) = \frac{X^{(2)}{}_1(i)}{\sqrt{\Sigma_2}}$	i
(10)	(11)	(12)	(13)	(14)	(15)	(16)	(1)
0.290	1.836	0.268	2.764	7.640	89.0	0.164	5
0.315	4.046	0.507	2.495	6.225	96.7	0.148	4
0.201	5.811	0.637	1.988	3.952	61.4	0.118	3
0.093	7.013	0.684	1.351	1.825	28.3	0.080	2
0.023	7.611	0.668	0.668	0.446	6.9	0.040	1
$\Sigma_1 = 0.922$					$\Sigma_2 = 282.3$		

MODE BY THE STODOLA–VIANELLO METHOD

$10^2\,X_2^{(2)}$	$M_i[X_2^{(2)}]^2$	$\bar{\phi}_2^{(2)}$	$\phi_2^{(2)}$	$M_i\phi_2^{(2)}$	$S_i^{(2)}$	$10^2\,\Delta X_2^{(2)}$	$10^2\,X_2^{(2)}$	$M_i[X_2^{(2)}]^2$	$\bar{\phi}_2^{(2)}$	$\phi_2^{(2)}$	i
(14)	(15)	(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(1)
−0.377	1.655	−0.162	−0.162	−1.885	−1.885	−0.276	−0.385	1.730	−0.164	−0.164	5
−0.106	0.175	−0.046	−0.046	−0.707	−2.592	−0.325	−0.110	0.187	−0.047	−0.047	4
0.199	0.615	0.085	0.085	1.326	−1.265	−0.139	0.215	0.719	0.092	0.092	3
0.354	1.950	0.152	0.152	2.362	1.097	0.107	0.354	1.944	0.151	0.151	2
0.258	1.031	0.111	0.111	1.718	2.814	0.247	0.247	0.947	0.105	0.105	1
	$\Sigma_2 = 5.426$							$\Sigma_2 = 5.527$			

equation, Eq. 43, becomes

$$\frac{49.60}{\Omega^2_k} - 1 = \frac{0.996}{57.11/\Omega^2_k - 1} + \frac{0.0024}{424.9/\Omega^2_k - 1}.$$

The first two roots of this equation are

$$\Omega^2{}_1 = 26.57; \quad \Omega_1 = 5.155$$
$$\Omega^2{}_2 = 421.1; \quad \Omega_2 = 20.52$$

Hence, by Eq. 41*b*, for $k = 1$:

$$c_{11} = \frac{0.998}{57.11/26.57 - 1}\, c_1 = 0.868\, c_1$$

and

$$c_{12} = \frac{-0.049}{424.9/26.57 - 1}\, c_1 = -0.003\, c_1$$

—and, by Eq. 42: $49.60\, c^2{}_1 + 57.11(0.868\, c_1)^2 + 424.9(-0.003\, c_1)^2 = 26.57$, from which $c_1 = 0.536$; $c_{11} = 0.465$; and $c_{12} = -0.002$. Repeating the computations for Ω_2: $c_2 = 0.184$; $c_{21} = -0.212$; and $c_{22} = -0.990$. Hence, the first two modes of the rocking structure are given by:

$$\psi_1(i) \doteq 0.536\, \phi_A(i) + 0.465\, \phi_1(i) - 0.002\, \phi_2(i)$$

and

$$\psi_2(i) \doteq 0.184\, \phi_A(i) - 0.212\, \phi_1(i) - 0.990\, \phi_2(i)$$

The shear modes $\psi_{k,s}$ of the rocking structure are evaluated by setting $\phi_A = 0$ in these equations. Table 6 contains the modes ψ_k and $\psi_{k,s}$ for $k = 1$ and $k = 2$.

c. Motion of Building on Rigid Soil for a Sinusoidal Foundation Displacement.—In order to evaluate the influence of the various modes on the motion of the building and to gage the convergence of the Noumerov-Fox step-by-step integration method, the building of Table 1 was subjected to a sinusoidal ground displacement of unit amplitude and period $T_0 = 1.2$ sec (the period of an important Fourier component of an actual earthquake record). With $\omega_0 = \frac{2\pi}{T_0} = 5.236$, and $\omega^2_0 = 27.42$, with ω_1 and ω_2 selected from Table 2, and ϕ_1 and ϕ_2 from Table 3(*a*), Eq. 24 gives the displacements:

$$x_i(t) = [1 + 7.318\ \phi_1(i) + 0.188\ \phi_2(i)] \sin \omega_0 t - 10.563\ \phi_1(i) \sin \omega_1 t - 0.741\ \phi_2(i) \sin \omega_2 t \ldots (70)$$

TABLE 7.—MOTION OF A BUILDING ON RIGID SOIL FOR A SINUSOIDAL FOUNDATION DISPLACEMENT

i	(*a*) SINE COEFFICIENTS OF $x_i(t)$ IN THE SOLUTION BY MODES					(*b*) SATISFACTION OF REST CONDITIONS				
	$\sin \omega_0 t$ (Mode 1)	$\sin \omega_0 t$ (Modes 1 and 2)	$\sin \omega_0 t$ (all modes)	$\sin \omega_1 t$	$\sin \omega_2 t$	r_1	r_2	r_3	r_4	r_5
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
5	2.204	2.173	2.178	−1.738	0.123	−0.304	0.149	−0.070	−0.015	0.000
4	2.087	2.078	2.076	−1.569	0.034	−0.177	−0.052	0.106	−0.040	0.000
3	1.866	1.884	1.878	−1.250	−0.070	0.062	−0.196	−0.026	0.084	0.000
2	1.588	1.616	1.617	−0.849	−0.111	0.303	−0.045	−0.155	−0.120	0.001
1	1.290	1.310	1.318	−0.419	−0.075	0.685	0.407	0.216	−0.100	0.000

Cols. 1, 2, and 3 of Table 7(*a*) contain, respectively, the coefficients of the forced term $\sin \omega_0 t$, when 1, 2, and 5 modes are taken into account in Eq. 24. (These last coefficients have been obtained independently from a rigorous solution of the same problem by the Laplace transform method.) These results indicate that in the present problem the contribution of the higher modes to the forced vibrations is negligible and that it is permissible to take into account only two modes. Cols. 4 and 5 of Table 7(*a*) contain the coefficients of $\sin \omega_1 t$ and $\sin \omega_2 t$, the contributions of the first two modes of free vibration.

It was noticed in Sect. 3 that the initial conditions of rest for the co-ordinates p_j cannot be rigorously satisfied unless all the modes are taken into account. Table 7(*b*) gives the values of the ratios $r_{N'}(i)$, from Eq. 20, for $N' = 1$, 2, 3, 4, 5, and shows that the rest conditions are very poorly satisfied at the lower floors unless one takes into account all the modes. This is to be expected, since the jump between $\dot{x}_0(0) \neq 0$ and $\dot{x}_i(0) = 0$ cannot be well approximated by a Fourier series. The values r_5, which should be identically zero, have been computed as a numerical check. It is noteworthy (and also lucky) that the solution of this problem is not too sensitive to the initial conditions.

d. The Step-by-step Integration of the Lagrangian Equations.—The displacements of the structure due to a sinusoidal motion of the ground were evaluated in Sect. 9c by means of the solution of Eqs. 16 given by Eqs. 23. The values of $p_1(t)$ and $p_2(t)$, for $X_0 = 1$ and $\omega_0 = 5.236$, in steps of 0.1 for p_1 and of 0.025 for p_2, appear in Table 8, for values of t between 0 and $t = 1.5\ T_0 = 1.8$ sec.

TABLE 8.—STEP-BY-STEP INTEGRATION OF THE LAGRANGIAN EQUATIONS FOR $p(t)$

r	Approximate p_r	True p_r	r	Approximate p_r	True p_r	r	Approximate p_r	True p_r	r	Approximate p_r	True p_r
(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)
(a) INTEGRATION OF p_1											
0	0	0	5	1.770	1.755	10	−2.963	−2.940	15	3.197	3.179
1	0.032	0.048	6	1.335	1.311	11	−2.173	−2.156	16	2.275	2.276
2	0.312	0.335	7	0.172	0.155	12	−0.465	−0.466	17	0.574	0.593
3	0.884	0.901	8	−1.351	−1.352	13	1.497	1.480	18	−1.173	−1.147
4	1.507	1.508	9	−2.599	−2.583	14	2.909	2.885	—	—	—
(b) INTEGRATION OF p_2											
1	0.004	0.006	19	0.752	0.748	37	−1.123	−1.118	55	0.870	0.866
2	0.041	0.044	20	0.749	0.744	38	−1.220	−1.214	56	1.081	1.075
3	0.135	0.138	21	0.682	0.676	39	−1.252	−1.245	57	1.236	1.229
4	0.293	0.295	22	0.537	0.533	40	−1.199	−1.192	58	1.311	1.304
5	0.505	0.505	23	0.320	0.318	41	−1.059	−1.055	59	1.296	1.291
6	0.744	0.742	24	0.054	0.054	42	−0.851	−0.849	60	1.202	1.199
7	0.974	0.970	25	−0.227	−0.224	43	−0.606	−0.606	61	1.056	1.056
8	1.157	1.151	26	−0.482	−0.477	44	−0.361	−0.364	62	0.894	0.895
9	1.265	1.258	27	−0.680	−0.674	45	−0.153	−0.157	63	0.750	0.753
10	1.284	1.278	28	−0.800	−0.795	46	−0.004	−0.008	64	0.652	0.654
11	1.221	1.217	29	−0.843	−0.839	47	0.078	0.076	65	0.610	0.610
12	1.099	1.096	30	−0.823	−0.823	48	0.106	0.106	66	0.615	0.614
13	0.951	0.950	31	−0.777	−0.777	49	0.107	0.109	67	0.645	0.641
14	0.813	0.815	32	−0.733	−0.735	50	0.113	0.117	68	0.666	0.660
15	0.716	0.718	33	−0.724	−0.726	51	0.158	0.162	69	0.643	0.637
16	0.673	0.675	34	−0.767	−0.769	52	0.261	0.264	70	0.553	0.548
17	0.680	0.680	35	−0.862	−0.862	53	0.426	0.427	71	0.387	0.383
18	0.717	0.715	36	−0.991	−0.989	54	0.639	0.638	72	0.154	0.154

The values of these functions at the same points have also been evaluated by the Noumerov-Fox step-by-step procedure. For $j = 1$, with $\omega^2_1 = 57.11$ and $h = 0.10$, Eqs. 62 give: $c_1 = 1.455$; and $F_{r,1} = 0.0454$ [$\sin 0.524\ (r + 1) + 10 \sin 0.524\ r + \sin 0.524\ (r - 1)$]; and Eqs. 67 give: $p_{1,1} = 0.317$; and $p_{2,1} = 0.312$. With these constants and starting values, Eq. 63 gives the values of $p_{r,1}$ in Table 8(*a*). A comparison with the true values indicates satisfactory agreement although h is about one eighth of the period $T_1 = \dfrac{2\pi}{7.557} = 0.831$. For $j = 2$, similarly: $c_2 = 1.740$; $F_{r,2} = 0.02165$ [$\sin 0.131\ (r + 1) + 10 \sin 0.131\ r + \sin 0.131\ (r - 1)$]; $p_{1,2} = 0.0041$; and $p_{2,2} = 0.0410$. By Eq. 63, the results are as shown in Table 8(*b*). These results agree satisfactorily with the true values after seventy-two steps, although the spacing h is of the order of $\frac{1}{12}$ the period $T_2 = 0.305$ sec.

The seventy-two steps in the integration of p_2 took approximately 20 min., once the sine functions had been tabulated. The error e_{r+1} was evaluated but

not used since it did not influence the results within the accuracy of the computations.

e. Influence of Damping.—It was stated in Sect. 4 that the influence of damping on the earthquake displacements consists, essentially, in reducing the importance of the higher modes. This is revealed by the case of the structure under study, if a logarithmic decrement equal to 0.9 is assumed, leading to a value of the damping coefficients (Eq. 29): $f_1 = 0.017 \times 7.557 = 0.127$; and $f_2 = 0.017 \times 20.61 = 0.350$. Solving Eqs. 27 with these values of the damping coefficient f_j, for a sinusoidal ground motion of unit amplitude and period $T = 1.2$ sec.—and using the modes $\phi_1(i)$ and $\phi_2(i)$ of the undamped structure of Table 3(*a*)—the floor displacements become

$$x_i(t) = \sin \omega_0 t + 7.311\, \phi_1(i) \sin (\omega_0 t - 2°31') + 0.190\, \phi_2(i) \sin (\omega_0 t - 8°8') - 10.568\, e^{-0.127t}\, \phi_1(i) \sin (\omega_1 t + 1°45') - 0.740\, e^{-0.350t}\, \phi_2(i) \sin (\omega_2 t - 2°5').$$

A comparison of this equation with Eq. 70 of Sect. 9*c* shows that the coefficients of the forced vibration are practically unchanged. The amplitude of the higher modes, instead, is rapidly damped out, and the frequencies are practically unchanged by friction.

10. Conclusions

The methods outlined in this paper permit the solution of dynamic problems of shear buildings whatever the applied forces or displacements may be. They must be compared with other methods on the basis of efficiency and flexibility.

A direct measure of their efficiency has been gained by the writer in solving the problem of the structure without rocking in three different ways:

(1) By direct integration of the equations of motion, using the Laplace transform;

(2) By numerical integration of the same equations, using an extension of the Noumerov-Fox method[15]; and

(3) By modes and numerical integration, using the methods of this paper.

The time involved in the three solutions was roughly in the ratio of 3:3:1; but this ratio of times does not indicate the main advantages of the use of modes—(a) the possibility of simplifying the solution by using a limited number of modes, and (b) the small amount of additional work necessary to analyze the structure under various types of external loads or displacements.

The solution of the rocking building by direct numerical integration is complicated[15] and requires 30 min for each step, whereas the use of complementary systems permits the evaluation of the rocking modes in a remarkably easy manner. The direct analytical integration of the same problem, whenever possible, is very cumbersome.

The introduction of internal damping does not present difficulties by the present methods, but complicates the other two types of solution considerably. It seems possible to state, therefore, that the methods presented herein may be

[15] "A Comprehensive Numerical Method for the Analysis of Earthquake Resisting Structures," by C. S. Whitney, B. G. Anderson, and M. G. Salvadori, *Journal*, A.C.I., Vol. 23, No. 1, September, 1951.

more efficient and flexible than others used to date. The availability of such procedures should encourage the designer to abandon the simplified and unrealistic methods of analysis suggested by codes (such as the $0.1g$ method) with obvious gains in economy and safety.

Finally, analogous methods are applicable to tall buildings, in which direct stresses are prevalent, and they may be extended to include the elasto-plastic behavior of structures under dynamic loads.

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